D2AMT1904 (S1) (2 PAGES) Reg. No. ……………

Name:………………

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021
(Improvement/Supplementary)

MATHEMATICS

FMTH2C09: ODE & CALCULAS OF VARIATIONS

Time: Three Hours **Maximum Weightage: 30**

Part A: Answer all questions. Each carries 1 weightage.

1. Classify and locate the singular points on the x axis of the differential equation $x(3x+1)y'' - (x+1)y' + 2y = 0$.
2. Determine the nature of the point at $x = \infty$ for the differential equation $x^2y'' + xy' + (x^2 - p^2)y = 0$
3. Show that $P_n(-x) = (-1)^n P_n(x)$, where $P_n(x)$ denote the n^{th} Legendre polynomial.
4. Find the Bessel series of the function $f(x) = 1$.
5. Show that the zeros of the functions $a\sin x + b\cos x$ and $c\sin x + d\cos x$ are distinct and occur alternately whenever $ad - bc \neq 0$.
6. Explain the different types of critical points of an autonomous system.
7. Define isoperimetric problem.
8. For what points (x_0, y_0) does Picard's theorem imply that the initial value problem $y' = y|y|$, $y(x_0) = y_0$ has a unique solution on some interval $|x - x_0| \leq h$?

(8 × 1 = 8 Weightage).

Part B: Answer any two questions from each unit. Each carries 2 weightage.

Unit I

9. Obtain the power series solution of the equation $(1-x^2)y'' - xy' + p^2y = 0$, where p is a constant.
10. Find the general solution of Gauss's Hypergeometric equation near its singular point $x = 0$.
11. Derive the Rodrigue's formula for Legendre polynomials.

Unit II

12. If $W(t)$ is the Wronskian of the two solutions of the homogeneous system

$$\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$$

then show that $W(t)$ is either identically zero or nowhere zero on $[a, b]$.

13. Determine the nature and stability properties of the critical point $(0, 0)$ for the system:

$$\begin{cases} \frac{dx}{dt} = 4x - 2y, \\ \frac{dy}{dt} = 5x + 2y. \end{cases}$$

14. Find the general solution of the following system:

$$\begin{cases} \frac{dx}{dt} = -4x - y, \\ \frac{dy}{dt} = x - 2y. \end{cases}$$

Unit III

15. Every nontrivial solution of $y'' + (\sin^2 x + 1)y = 0$ has an infinite number of positive zeros. Formulate and prove a theorem that includes this statement as a special case.

16. Obtain Euler's differential equation for an extremal.

17. Check whether $f(x, y) = xy^2$ satisfies a Lipshitz condition on any rectangle $a \leq x \leq b$ and $c \leq x \leq d$.

(6 $\times$ 2 = 12 Weightage).

Part C: Answer any two questions. Each carries 5 weightage.

18. a) Solve the ordinary differential equation $(1 + x)y' = py$ with the condition $y(0) = 1$.
    b) State and prove the orthogonality property of Legendre polynomials.

19. Find Frobenius series solution of the equation $xy'' - y' + 4x^3y = 0$.

20. Find the general solution of Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$, when p is not an integer.

21. Explain the Picard's method of successive approximations to solve second order linear differential equation. Solve the following initial value problem by Picard's method:

$$\begin{cases} \frac{dy}{dx} = z, & y(0) = 1, \\ \frac{dz}{dx} = -y, & z(0) = 0. \end{cases}$$

(2 $\times$ 5 = 10 Weightage).