(2 Pages)

D1AMT2403 **Reg. No:** ………………….
Name: …………………………….

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEBER 2024
MATHEMATICS
FMTH1C03: Real Analysis I

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage

1. Construct a bounded subset of real numbers with exactly three limit points.
2. Is R compact? Justify your answer.
3. Show that continuous image of a connected set is connected.
4. State mean value theorem. Give an example to show that the mean value theorem fails to hold for complex valued functions.
5. If $f(x)=0$ for all irrational x and $f(x)=1$ for all rational x, prove that f is not Riemann integrable on $[a,b]$ for any $a<b$.
6. If $f\in\Re(\alpha)$ on $[a,b]$ then prove that $|f|\in\Re(\alpha)$ and $\left|\int_a^b f\,d\alpha\right|\le\int_a^b |f|\,d\alpha.$
7. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
8. If $\{f_n\}$ is a sequence of continuous complex valued functions on a compact metric space K, and if $\{f_n\}$ converges uniformly on K, then show that $\{f_n\}$ is equicontinuous on K.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage

Unit 1

9. Show that a set is open if and only if its complement is closed.
10. Show that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.
11. Prove that a continuous mapping f of a compact metric space X into a metric space Y is uniformly continuous.

Unit 2

12. Suppose f is a real differentiable function on $[a,b]$ and suppose that $f'(a) < \lambda < f'(b)$, then prove that there exists a point $x \in (a,b)$ such that $f'(x) = \lambda$.

13. If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$, then show that $f \in \Re(\alpha)$.

14. If $f \in \Re$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then prove that $\int_a^b f(x)\,dx = F(b) - F(a)$.

Unit 3

15. If γ' is continuous on $[a,b]$, then show that γ is rectifiable, and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$.

16. If $\{f_n\}$ is a sequence of continuous functions on E and if $f_n \to f$ uniformly on E, then prove that f is continuous on E.

17. If $\{f_n\}$ is a sequence of continuous complex valued functions on a compact metric space K, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then show that $\{f_n\}$ is uniformly bounded on K, and $\{f_n\}$ contains a uniformly convergent subsequence.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage

18. (a) Prove that the set of all sequences whose elements are the digits 0 and 1 is uncountable.

    (b) Prove that every k–cell is compact.

19. (a) If E is a non-compact set of real numbers, prove that there exists a continuous and bounded function on E which has no maximum.

    (b) Show that monotone functions have no discontinuities of the second kind.

20. (a) Suppose f and g are real and differentiable in (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$. Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or if $g(x) \to +\infty$ as $x \to a$, then show that $\frac{f(x)}{g(x)} \to A$ as $x \to a$.

(b) Prove that $f \in \Re(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

21. (a) If f is a continuous complex function on $[a,b]$, prove that there exists a sequence of polynomials P_n such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$.

(b) For every interval $[-a, a]$, show that there exist a sequence of real polynomials P_n such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a, a]$.

(2 × 5 = 10 weightage)