D1AMT2502 (3 Pages) Name: ....
Reg. No:.....................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2025

(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH1C02- LINEAR ALGEBRA

Time: 3 Hours **Maximum: 30 Weightage.**

Part A: Answer ***all*** questions. Each carries 1 weightage.

1. Let V be a vector space over the field F. Prove that the intersection of any collection of subspaces of V is a subspace of V.
2. Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$defined by $T(x_1, x_2) = (\sin x_1, x_2)$. Verify whether T is a linear transformation. Justify your answer.
3. Let F be a field and let T be the linear operator on F^2 defined by $T(x_1, x_2) = (x_2, x_1+x_2)$. Then prove that T is non-singular, also find T^{-1}.
4. Let$F$ be a field and let f be the linear functional on F^2 defined by $f(x_1, x_2) = ax_1 + bx_2$. Let $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$. Find $T^t f$
5. Suppose that $T\alpha = c\alpha$. If f is any polynomial, then prove that $f(T)\alpha = f(c)\alpha$.
6. Define an invariant subspace of a vector space V.Prove that for a linear transformation T, Null space of T is invariant under T.
7. Let $\alpha = (1, 2) \quad \beta = (-1, 1)$. If γ is a vector such that $(\alpha|\gamma) = -1$ and $(\beta|\gamma) = 3$. Find γ.
8. Prove that an orthogonal set of non-zero vectors is linearly independent.

. **(8 x1=8 weightage)**

Part B: Answer ***any two*** questions from ***each unit***. *Each carries 2 weightage.*

UNIT I

9. Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then prove that any independent set of vectors in V is finite and contains no more than m elements. .

(P.T.O.)

10. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the ordered basis for $\mathbb{R}^4$ consisting of $\alpha_1 = (1, 1, 0, 0)$, $\alpha_2 = (0, 0, 1, 1), \alpha_3 = (1, 0, 0, 4), \alpha_4 = (0, 0, 0, 2)$. Find the coordinates of each of the standard basis vectors in the ordered basis $\mathcal{B}$.

11. Let $V$and W be vector space over the field F and let T be a linear transformation from V into W. Suppose that $V$is finite dimensional. Then prove that:
rank(T)+ nullity(T)= $\dim V$.

UNIT II

12. Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_2 - x_3)$. IF $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{B}' = \{\beta_1, \beta_2\}$, where $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1), \alpha_3 = (1, 0, 0), \beta_1 = (0, 1), \beta_2 = (1, 0)$. Find the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.

13. Let V be a finite dimensional vector space over the field F. For each vector αin V define $L_\alpha(f) = f(\alpha)$,$f \in V^*$.Then prove that the mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}.

14. Let T be a linear operator on a finite dimensional space V. Let $c_1, c_2, \cdots c_k$ be the distinct characteristic values of T and let W_i be the null spaces of $(T - c_i I)$. Then prove that following are equivalent

    (a) T is diagonalizable

    (b) The characteristic polynomial for T is $f = (x - c_1)^{d_1}(x - c_2)^{d_2} \cdots (x - c_k)^{d_k}$ and $\dim W_i = d_i; i = 1, 2, \ldots k$.

    (c) dim W_1+ dim W_2+ $\cdots$ dim W_k= dim V.

UNIT III

15. Let V be a finite -dimensional vector space. Let $W_1, W_2, \ldots, W_k$ be subspaces of V and let $W = W_1 + W_2 + \ldots + W_k$. Then following are equivalent.

    (a) $W_1, W_2, \ldots W_k$ are independent.

    (b) For each j, $2 \leq j \leq k$, we have $W_j \cap (W_1 + \ldots W_{j-1}) = \{0\}$

    (c) If $\mathcal{B}_i$is an ordered basis for W_i $1 \leq i \leq k$, then the sequence $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2 \ldots \mathcal{B}_k)$ is an ordered basis for W.

16. (a) Define an inner product on a vector space V.

    (b) State and Prove Cauchy- Schwarz inequality on an inner product space V.

17. Apply the Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1), \beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

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(6 x2=12 weightage)

Part C: Answer **any two** questions. Each carries 5 weightage.

18. (a) Let V be an n- dimensional vector space over the field F, and let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered bases of V. Then prove that there is a unique, necessarily invertible $n \times n$ matrix P with entries in F such that $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$ and $[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$.

    (b) Prove that a linear transformation is nonsingular if and only if it is one-one.

19. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If V and W are finite dimensional then prove the following.

    (a) rank(T^t)= rank(T)

    (b) The range of T^t is the annihilator of the null space of T^t.

20. State and Prove Cayley-Hamilton theorem.

21. (a) Let E be the orthogonal projection of V on W. Then prove that E is an idempotent linear transformation of V onto W, $W^\perp$ is the null space of E, and $V = W \oplus W^\perp$.

    (b) State and prove Bessel's inequality.

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(2 x 5=10 weightage)