

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2025
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH1C01-ABSTRACT ALGEBRA

Time: 3 Hours

Maximum Weightage: 30

Part A: Answer all questions. Each carries 1 weightage.

1. Find the order of $(20,13,15)$ in the group $\mathbb{Z}_{25} \times \mathbb{Z}_{52} \times \mathbb{Z}_{20}$.
2. Prove that factor group of a cyclic group is cyclic.
3. Find the number of orbits in $\{1,2,3,4,5,6,7,8,9,10\}$ under the cyclic subgroup $\langle (1,2,3) \rangle$ of S_{10} .
4. Let $\phi: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_3$ be the homomorphism such that $\phi(1) = 2$. Find the kernel of ϕ .
5. Let G be a finite group and let $|G| = p^5 m$, where p is a prime and p is not a divisor of m . Prove or disprove there exists a subgroup H of G such that $|H| = p^2$.
6. Give a group presentation of S_3 involving two generators.
7. Prove that $x^2 - 2$ has no zeros in the rational numbers.
8. Define an ideal of a ring. Give an example.

(8 × 1 = 8 weightage)

Part B: Answer any two questions from each unit. Each carries 2 weightage.

Unit 1

9. State fundamental theorem of finitely generated abelian groups. Find all abelian groups, up to isomorphism, of order 725.
10. Let N be a subgroup of a group G . Prove that left coset multiplication is well defined by the equation $(aN)(bN) = (ab)N$ if and only if $gN = Ng$ for all $g \in G$.
11. Let G be a group and let M be a normal subgroup of G . Prove that the factor group $\frac{G}{M}$ is simple if and only if M is maximal.

Unit 2

12. Let H be a subgroup of G and let N be a normal subgroup of G . Prove that the factor groups $\frac{HN}{N}$ and $\frac{H}{H \cap N}$ are isomorphic.
13. Let G be a finite group and let P_1 and P_2 be Sylow p - subgroups of G . Prove that $P_1 = gP_2g^{-1}$ for some $g \in G$.
14. Let G be a group of order p^r , where p is prime. Prove that G is solvable.

(P.T.O.)

Unit 3

15. Let F be a field and let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$ be two elements of $F[x]$, with a_n and b_m both nonzero elements of F and $m > 0$. Then prove that there are unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$, where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree m of $g(x)$.
16. Let p be a prime. Prove that $x^{p-1} + x^{p-2} + \dots + x + 1$ is irreducible over \mathbb{Q} .
17. Let F be a field. Prove that every ideal in $F[x]$ is principal.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. a) Let X be a G -set and let $x \in X$. Prove that $|Gx| = (G : G_x)$.
- b) Find the number of distinguishable ways the edges of a square of cardboard can be painted if six colors of paint are available and the same color can be used on any number of edges.
19. a) Let $\phi: G \rightarrow G'$ be an onto group homomorphism with kernel H . Prove that $\frac{G}{H}$ and G' are isomorphic.
- b) Let G be a group. Prove that two normal series of G have isomorphic refinements.
20. a) Let G be a finite group of order m and let p be a prime factor of m . Suppose that G has n Sylow p -subgroups. Prove that (i) n is a divisor of m . (ii) $n \equiv 1 \pmod{p}$.
- b) Prove that no group of order 96 is simple.
21. a) Let $p \in \mathbb{Z}$ be a prime. Suppose that $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ is in $\mathbb{Z}[x]$ $a_n \not\equiv 0 \pmod{p}$, but $a_i \equiv 0 \pmod{p}$ for all $i < n$, with $a_0 \not\equiv 0 \pmod{p^2}$. Prove that $f(x)$ is irreducible over \mathbb{Q} .
- b) Let F be a field and let $\langle p(x) \rangle$ be a non trivial ideal of $F[x]$. Prove that $\langle p(x) \rangle$ is maximal if and only if $p(x)$ is irreducible over F

(2 × 5 = 10 weightage)