(2 Pages)

D1AMT2405 **Name............................**

Reg. No. ....

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2024
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH1C05: NUMBER THEORY

Time : 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Prove or disprove: "$a|b$ if and only if $\phi(a)|\phi(b)$". Justify.
2. State Abel's identity.
3. Show that for all $x \geq 1$, $\sum_{n\leq x} \theta(\frac{x}{n}) = x \log x + O(x)$.
4. Prove that for $n \geq 1$, $\sum_{d/n} \phi(d) = n$.
5. Find the inverse of the matrix $\begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}$ *(mod 5)*.
6. Find all integers n such that $\phi(n) = n/2$.
7. Does there exist a function which is multiplicative but not completely multiplicative? Justify.
8. Prove that if $2^n + 1$ is a prime, then n is a power of 2.

(8 x 1 = 8 weightage)

Part B: Answer *any two* questions from *each unit*. Each carries 2 weightage.

Unit 1

9. State and prove Legendre's identity.
10. Prove that for $n \geq 1$, $\sum_{d/n} \Lambda(d) = \log n$.

(P.T.O.)

11. Show that for $n \geq 1$, $\sum_{d/n} \mu(d) = [\frac{1}{n}]$ where μ denotes the Mobius function.

Unit 2

12. Show that for $n \geq 1$, $\frac{1}{6} n \log n < p_n < 12(n \log n + n \log \frac{12}{e})$ where p_n denotes the n^{th} prime.

13. Prove that $\lim_{x\to\infty}(\frac{M(x)}{x} - \frac{\mu(x)}{x \log x}) = 0$.

14. Show that $\lim_{x\to\infty} \frac{\pi(x) \log x}{x} = 1$ is logically equivalent to $\lim_{x\to\infty} \frac{\pi(x) \log \pi(x)}{x} = 1$.

Unit 3

15. Show that $(n|p)$ is a completely multiplicative function of n.

16. Determine whether 888 is a quadratic residue modulo 1999.

17. Using the 2×2 enciphering matrix $A = \begin{pmatrix} 2 & 3 \\ 7 & 8 \end{pmatrix}$ with a 26-letter alphabet, decipher the ciphertext ”FWMDIQ”.

(6 x 2 = 12 weightage)

Part C: Answer *any two* questions. Each carries 5 weightage.

18. a) State Euler's summation formula.
    b) Show that for $x \geq 1$, $\sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$ if $\alpha \geq 0$.
    c) Show that for $x \geq 2$, $\log[x]! = x \log x - x + O(\log x)$.

19. Prove that for $n \geq 2$, $\frac{1}{6} \frac{n}{\log n} < \pi(n) < 6 \frac{n}{\log n}$.

20. State and prove Euler's criterion.

21. State and prove Quadratic reciprocity law for Legendre symbol.

(2 x 5 = 10 weightage)