D1AMT2403 **(3 Pages)** **Name.............................**
Reg.No..........................

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2024
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH1C03-REAL ANALYSIS I

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Define a metric space. Let X be an infinite set. For $p \in X$ and $q \in X$, define

   $d(p,q) = \begin{cases} 1, & if\ p \neq q \\ 0, & if\ p = q \end{cases}$. Prove that d is a metric.

2. Define an interior point, open set and a neighborhood. Prove that every neighborhood is an open set.

3. Let A be the set of all sequences whose elements are 0 and 1. Prove that A is uncountable.

4. Discuss the continuity/discontinuity behavior of the function f defined by

$$f(x) = \begin{cases} 1, & if\ x\ is\ rational \\ 0, & if\ x\ is\ irrational \end{cases}$$

5. Let f be defined for all real x and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all real x and y. Prove that f is a constant.

6. If K is a compact metric space and $f_n \in \mathcal{C}(K)$ for $n = 1,2,3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then prove that $\{f_n\}$ is equicontinuous on K.

7. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

8. State Stone-Weierstrass Theorem and define a sequence of real polynomials P_n in $[a, b]$ such that $P_n(0) = 0$ and $\lim_{n\to\infty} P_n(x) = |x|$.

(8 × 1= 8 weightage)

(P.T.O.)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage

Unit 1

9. If X is a metric space and $E \subset X$, then prove the following.

   a. $\bar{E}$ is closed.
   b. $E = \bar{E}$, if and only if E is closed.
   c. $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

10. Define a compact set and give an example. Prove that compact subsets of metric spaces are closed.

11. Define a connected set and give an example. Let E be a subset of the real line R^1. Prove that E is connected if and only if it has the property: If $x \in E,\ y \in E$ and $x < z < y$, then $Z \in E$.

Unit 2

12. Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$ where $a \leq t \leq b$, then prove that h is differentiable at x and $h'(x) = g'(f(x))f'(x)$. Using this find the derivative of the function

    $h(x) = tan^{-1}(\sqrt{\cos x})$.

13. Define Riemann Stieltjes integral. If $f_1(x) \leq f_2(x)$ on $[a, b]$, then prove that $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$.

14. Show that f' need not be continuous even if f is a differentiable function using an example.

Unit 3

15. If γ' is continuous on $[a, b]$, then prove that γ is rectifiable and $\wedge(\gamma) = \int_a^b |\gamma'(t)| dt$.

16. If K is a compact metric space and $f_n \in \mathcal{C}(K)$ for $n = 1,2,3, ...$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then prove that $\{f_n\}$ contains a uniformly convergent subsequence.

17. Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathfrak{R}(\boldsymbol{\alpha})$ on $[a, b]$**,** for n = 1,2,3,…, and suppose $f_n \to f$ uniformly on $[a, b]$. Then prove that $f \in \mathfrak{R}(\boldsymbol{\alpha})$ on [a,b] and $\int_a^b f dx = \lim_{n \to \infty} \int_a^b f_n dx$.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then prove that f is uniformly continuous on X.

19. Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is continuous at every point at which f is discontinuous. Then prove that $f \in \mathfrak{R}(\alpha)$.

20. State and prove the fundamental theorem of Calculus.

21. Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that $\lim_{t \to x} f_n(t) = A_n$ for n=1,2,3,… Then prove that $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n\to\infty} A_n$. Check whether the sequence $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$, $0 \le x \le 1$, $n = 1,2,3, \ldots$ is uniformly convergent.

(2 × 5 = 10 weightage)