D1AMT2402 (2 Pages) Name:................................
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FIRST SEMESTER M Sc. DEGREE EXAMINATION, NOVEMBER 2024
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH1C02: LINEAR ALGEBRA

Time : 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Let V be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let V_e and V_o be the subsets of all even functions and odd functions from $\mathbb{R}$ to $\mathbb{R}$ respectively. Show that V_e is a subspace of V. Can you say V_o is a subspace of V? Give reason.
2. For which value of c will the vector $\alpha = (11, 8, c)$ in $\mathbb{R}^3$ be a linear combination of the vectors $a = (1, 2, 2)$ and $b = (5, 3, 7)$?
3. Show that the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $f(x, y, z) = (2x^2, x - y)$ is not linear.
4. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear map defined by $T(x, y) = (2x + 3y, x - y)$. Then Compute $[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the standard orderd basis of $\mathbb{R}^2$.
5. Let W_1 and W_2 be the subspace of a finite dimensional vector space. Prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.
6. Let λ be an eigen value of an operator $T : V \longrightarrow V$. Show that the eigen space of λ is a subspace of V.
7. Define a minimal polynomial for a linear operator T.
8. Compute the Standard inner product of $(1, 2, 3)$ and $(4, 5, 6)$ in $\mathbb{R}^3$.

. **(8 x 1= 8 weightage)**

Part B: Answer *any two* questions from *each unit*. Each carries 2 weightage.

Unit 1

9. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis consisting of $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 = (1, 0, 0)$. What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?
10. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear transformation with $T(1, 2) = (3, 2, 1)$ and $T(3, 4) = (6, 5, 4)$. Compute $T(0, 1)$.
11. Let V be an $n-$ dimensional and W be an $m-$ dimensional vector space over F. Then prove that $L(V, W)$ is finite dimensional and has dimension mn.

(P.T.O.)

Unit 2

12. Let V be a finite dimensional vector space over F and let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\}$ and $\mathcal{B}' = \{\alpha_1', \alpha_2', \alpha_3', \cdots, \alpha_n'\}$ be ordered basis for V. Suppose T is a linear operator on V. Prove that there exists an invertible matrix P such that $[T]_{\mathcal{B}}' = P^{-1}[T]_{\mathcal{B}}P$.

13. Let T be a linear operator on an $n-$ dimensional vector space V. Then prove that the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

14. Let V be a finite dimensional vector space over F and T be a linear operator on V. Then prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p(x) = (x - c_1)(x - c_2)(x - c_3)\cdots(x - c_n)$, where $c_1, c_2, c_3, \cdots c_n$ are distinct elements of F.

Unit 3

15. Let $E_1, E_2, E_3, \cdots, E_k$ are k linear operators on V, which satisfy:
    (a) each E_i is a projection
    (b) $E_iE_j = 0$, if $i \neq j$
    (c) $I = E_1 + E_2 + E_3 + \cdots + E_k$
    (d) The range of $E_i = W_i$

    Then Prove that $V = W_1 \oplus W_2 \oplus W_3 \oplus \cdots \oplus W_k$.

16. State and prove Bessel's inequality.

17. Verify that the following is an inner product in $\mathbb{R}^2$.
    $\langle \alpha, \beta \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$, where $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$.

(6 x 2= 12 weightage)

Part C: Answer *any two* questions. Each carries 5 weightage.

18. State and prove Rank Nullity theorem.

19. Let V and W be vector spaces over F with $\{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\}$ an ordered basis for V. Let $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ be any vectors in W. Prove that there exists a unique linear trasformation $T : V \longrightarrow W$ with $T(\alpha_k) = \beta_k$, for $k = 1, 2, 3, \cdots, n$.

20. Let W be the subspace of $\mathbb{R}^5$ spanned by $(2, -2, 3, 4, -1), (-1, 1, 2, 5, 2), (0, 0, -1, -2, 3)$ and $(1, -1, 2, 3, 0)$. Find a basis for W^0.

21. State Gram- Schmidt orthogonalization process. Apply Gram- Schmidt orthogonalization process to the vectors $(3, 0, 4), (-1, 0, 7)$ and $(2, 9, 11)$ in $\mathbb{R}^3$.

(2 x 5= 10 weightage)