D1AMT2401 (2 Pages) **Name: ....**

Reg. No. ....

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2024
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH1C01-ABSTRACT ALGEBRA

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Find the order of $(4, 2)$ in $Z_{12} \times Z_8$.
2. Find the number of Sylow 5 - subgroups of a group G where $|G| = 75$.
3. Find all abelian groups upto isomorphism of order 32.
4. Does every group of order 42 have a normal subgroup of order 7 ? Justify.
5. Is $8x^3 + 6x^2 - 9x + 24$ is irreducible over Q ? Justify.
6. Find the center of $Z_3 \times S_3$.
7. Find a prime ideal of $Z \times Z$ that is not a maximal ideal.
8. Find all $c \in Z_3$ such that $Z_3[x]/ < x^2 + c >$ is a field.

(8 x 1 = 8 weightage)

Part B: Answer *any two* questions from *each unit.* Each carries 2 weightage.
Unit 1

9. Prove that if m divides the order of a finite abelian group G, then G has a subgroup of order m.
10. Let X be a G-set and $x \in X$. Then prove that $|Gx| = (G : G_x)$. Also prove that if $|G|$ is finite,then $|G_x|$ is a divisor of $|G|$.
11. Let H be a subgroup of a group G. Prove that the left coset multiplication is well defined by the equation $(aH)(bH) = (ab)H$ if and only if H is a normal subgroup of G.

(P.T.O.)

Unit 2

12. (a) Prove that for a prime number p, every group G of order p^2 is abelian.
(b) Give isomorphic refinement of the two series $0 < 10Z < Z$ and $0 < 25Z < Z$.

13. Prove that every group of order 255 is abelian and cyclic.

14. Let p be prime. Let G be a finite group and let p divides $|G|$. Then prove that G has an element of order p and consequently a subgroup of order p.

Unit 3

15. Prove that the polynomial ϕ_p(x)= $x^{p-1} + x^{p-2} + ... + x + 1$ is irreducible over Q for any prime p.

16. Prove that an ideal $< p(x) > \neq 0$ of $F[x]$ is maximal if and only if p(x) is irreducible over F.

17. If G is a finite subgroup of the multiplicative group $< F^*, . >$ of a field F, then prove that G is cyclic.

(6 x 2 = 12 weightage)

Part C: Answer *any two* questions. Each carries 5 weightage.

18. Let R be a commutative ring with unity. Then prove that M is a maximal ideal of R if and only if R/M is a field.

19. (a) State and prove Third Sylow theorem.
(b) If p and q are distinct primes with $p < q$, then prove that every group G of order pq has a single subgroup of order q and this subgroup is normal in G. Also prove that if $q \not\equiv (1$ $modp)$ then G is abelian and cyclic.

20. (a) Prove that the group $Z_m \times \mathrm{Z}_n$ is cyclic and is isomorphic to Z_{mn} if and only if m and n are relatively prime.
(b) State the Fundamental theorem of finitely generated abelian groups.

21. (a) Prove that every group is the homomorphic image of a free group.
(b) Determine all groups of order 21 upto isomorphism.

(2 x 5 = 10 weightage)