D1AMT2305 **(2 Pages)** **Name......................**

Reg.No...................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023

(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH1C05-NUMBER THEORY

Time: 3 Hours **Maximum weightage: 30**

Part A: Answer *all* questions. Each carries *1* weightage.

1. Find all integers n such that $\phi(n) = \frac{n}{2}$.
2. Prove that $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ is 0 or 1.
3. Define Chebyshev's ψ and ϑ - function and obtain the relation between them.
4. For any Arithmetic functions α and β, prove that $\alpha \circ (\beta \circ F) = (\alpha \star \beta) \circ F$.
5. Prove that the value of the Euler totient function $\phi(n)$ is even for $n \geq 3$.
6. For $x \geq 2$, prove that $\pi(x) = \frac{\vartheta(x)}{logx} + \int_2^x \frac{\vartheta(t)}{tlog^2 t} dt$.
7. Find Legendre's symbol (73/383).
8. State and prove Euler's criterion.

(8 x 1= 8 weightage)

Part B

Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. If $n \geq 1$, prove that $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.
10. Prove that if f is multiplicative, then $\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p))$ and hence prove that $\phi^{-1}(n) = \prod_{p|n} (1 - p)$.
11. State and Prove Euler Summation Formula.

Unit 2

12. State and prove Abel's Idenity.
13. If for $x \geq 1$, $M(x) = \sum_{n \leq x} \mu(x)$, prove that $\lim_{x \to \infty} \frac{M(x)}{x} = 0$.

(P.T.O.)

14. For $x > 0$, Prove that, $0 \leq \frac{\psi(x)}{x} - \frac{\jmath(x)}{x} \leq \frac{(logx)^2}{2\sqrt{x}log2}$.

Unit 3

15. In a 27 letter alphabet system(with blank as 26) using affine enciphering transformation with key $a = 13$, $b = 98$, encipher the message ' HELP ME'.

16. State and prove Quadratic reciprocity law.

17. Prove that Legendre's symbol (n/p) is a complete multiplicative function of n.

(6 x 2= 12 weightage)

Part C

Answer any *two* questions. Each carries 5 weightage.

18. State and prove Gauss Lemma.

19. (a) Prove that if g and $f \star g$ are multiplicative, f is multiplicative.

    (b) Let f be multiplicative. Prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n)f(n); \forall n \geq 1$.

20. For $n \geq 2$, Prove that $\frac{1}{6}\frac{n}{logn} < \pi(n) < 6\frac{n}{logn}$.

21. Let P_n denote the n^{th} prime then, prove that the following relations are logically equivalent.

    (a) $\lim\limits_{x \to \infty} \dfrac{\pi(x)logx}{x} = 1$

    (b) $\lim\limits_{x \to \infty} \dfrac{\pi(x)log\pi(x)}{x} = 1$

    (c) $\lim\limits_{n \to \infty} \dfrac{P_n}{nlogn} = 1$

(2 x 5= 10 weightage)