D1AMT2303 (2 Pages) Name..............................
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FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2023
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH1C03: REAL ANALYSIS I

Time: 3 Hours Maximum Weightage: 30

Part A: Answer *all* questions. Each carries *1* weightage.

1. Prove that open balls are convex subsets of $\mathbb{R}^k$.
2. Construct a bounded set of real numbers with exactly three limit points.
3. If f is a continuous mapping of a metric space X into a metric space Y, then prove that
$$f(\overline{E}) \subset \overline{f(E)}.$$
4. True or False: Every continuous function is uniformly continuous. Substantiate your claim.
5. State Cauchy criterion for uniform convergence.
6. Define rectifiable curves.
7. Give an example of a function which is not Riemann- Stieltjes integrable. Justify your answer.
8. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from *each unit.* Each carries *2* weightage.

Unit 1

9. Prove that every infinite subset of a countable set A is countable.
10. Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.
11. Prove that monotone functions have no discontinuities of the second kind.

(P.T.O.)

Unit 2

12. Prove that $f \in R(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

13. (a) Suppose f is a real differentiable function on $[a, b]$ such that $f'(a) < \lambda < f'(b)$. Then prove that there exists $x \in (a, b)$ such that $f'(x) = \lambda$.

(b) State and prove fundamental theorem of calculus.

14. If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then prove that $\int_a^b f \, d\alpha = f(s)$.

Unit 3

15. Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that $\lim\limits_{t \to x} f_n(t) = A_n$. Then prove that $\lim\limits_{t \to x} f(t) = \lim\limits_{n \to \infty} A_n$.

16. Prove that $\mathscr{C}(X)$ is a complete metric space.

17. If K ia compact metric space, if $f_n \in \mathscr{C}(X)$ for $n = 1, 2, \ldots$ and if $\{f_n\}$ converges uniformly on K, then prove that $\{f_n\}$ is equicontinuous on K.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries *5* weightage.

18. (a) Prove that a nonempty perfect set in $\mathbb{R}^k$ is uncountable.

(b) Prove that every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

19. (a) Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then prove that $f(X)$ is compact.

(b) State and prove L'Hospital$'$s Rule.

20. (a) Assume α increases monotonically and α' is Riemann integrable on $[a, b]$. Then prove that a bounded real function f on $[a, b]$ is Riemann-Stieltjes integrable if and only if fα' is Riemann integrable on $[a, b]$.

(b) State and prove integration by parts rule.

21. State and prove Stone- Weierstrass theorem.

(5 × 2 = 10 weightage)