D1AMT2302 **(2 Pages)** **Name............................**

Reg.No..........................

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2023
(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH1C02 - LINEAR ALGEBRA

Time: 3 hours Maximum weightage: 30

Part A: Answer *all* questions. Each carries *1* weightage.

1. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Show that there exists unique vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2 \ \forall \ \alpha \in V$.
2. Find all c such that $\{(1, 2, 3), (1, 3, 1), (0, c, c)\}$ is a basis for $\mathbb{R}^3$.
3. Let V be a vector space of all 2×2 matrices over the field F. Show that $\dim_F(V) = 4$ by providing a basis for V.
4. Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$? Justify your answer.
5. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + 4x_3, -x_1 + x_2, -x_1 + 3x_2 + 4x_3).$$

   Find the matrix of T relative to the standard ordered basis of $\mathbb{R}^3$.
6. Show that the trace function is a linear functional on the matrix space $F^{m\times n}$.
7. Prove that any projection on a vector space is diagonalizable.
8. Define inner product on a vector space and show that if $(\alpha/\beta) = 0 \ \forall \ \beta \in V$, then $\alpha = 0$.

(8 x 1 = 8 weightage)

Part B: Answer *any two* questions from each unit.
Each carries *2* weightage.

Unit 1

9. Let $\mathcal{B} = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ be the ordered basis for $\mathbb{R}^3$. What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?
10. Let T be a linear transformation from V into W. Prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

(P.T.O.)

11. Prove that every $n-$dimensional vector space over the field F is isomorphic to the space F^n.

Unit II

12. If W is a $k-$dimensional subspace of an $n-$dimensional vector space V, prove that W is the intersection of $(n-k)$ hyperspaces in V.

13. Let a, b and c be elements of a field F, and let A be the following 3×3 matrix over F:

$$A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$$

Prove that the characteristic polynomial for A is $x^3 - ax^2 - bx - c$ and that this is also the minimal polynomial for A .

14. Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

Unit III

15. Let T be a linear operator on a finite-dimensional vector space V. Let R be the range of T and let N be the null space of T. Prove that R and N are independent if and only if $V = R \oplus N$.

16. State and prove the Cauchy-Schwarz inequality.

17. Prove that every finite-dimensional inner product space has an orthonormal basis.

(6 x 2 = 12 weightage)

Part C: Answer *any two* questions. Each carries *5* weightage.

18. Let W_1 and W_2 be two finite-dimensional subspaces of a vector space V, then prove that $\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

19. State and prove Rank-Nullity theorem.

20. State and prove the Cayley-Hamilton Theorem.

21. Consider $\mathbb{R}^4$ with the standard inner product. Let W be the subspace of $\mathbb{R}^4$ consisting of all vectors which are orthogonal to both $\alpha = (1, 0, -1, 1)$ and $\beta = (2, 3, -1, 2)$. Find a basis for W.

(2 x 5 = 10 weightage)