D1AMT2301 **(2 Pages)** **Name............................**

Reg.No..........................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2023
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH1C01 - ABSTRACT ALGEBRA

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries *1* weightage.

1. Find the order of the element $(2,1)+\langle(1,1)\rangle$ in $(\mathbb{Z}_3\times\mathbb{Z}_6)/\langle(1,1)\rangle$.
2. Let X be a G-set. Show that $G_x=\{g\in G \mid gx=x\}$ is a subgroup of G for each $x\in X$.
3. How many distinguishable necklaces (with no clasp) can be made using seven different colored beads of the same size.
4. Show that the group S_3 is solvable.
5. If p and q are distinct primes with $p<q$, show that every group G of order pq is not simple.
6. Show that both $\{1\}$ and $\{2,3\}$ are bases for $\mathbb{Z}_6$.
7. Find all zeros of $x^5+3x^3+x^2+2x$ in $\mathbb{Z}_5$.
8. Define group algebra and construct the addition and multiplication table for the group algebra $\mathbb{Z}_2G$ where $G=\{e,a\}$ is cyclic group of order 2.

(8 x 1 = 8 weightage)

Part B: Answer *any two* questions from each unit. Each carries *2* weightage.

Unit 1

9. If m divides the order of a finite abelian group G, prove that G has a subgroup of order m.
10. Prove that M is a maximal normal subgroup of G if and only if G/M is simple.
11. State and prove Burnside's formula.

Unit 2

12. Let H be a subgroup of G and let N be a normal subgroup of G. Prove that: $(HN)/N\simeq H/(H\cap N)$.
13. Let H be a p-subgroup of a finite group G. Prove that: $(N[H]:H)\equiv(G:H)(\text{mod }\ p)$.
14. Prove that no group of order 96 is simple.

(P.T.O.)

Unit 3

15. If G is a finite subgroup of the multiplicative group $\langle F^*, .\rangle$ of a field F, show that G is cyclic.

16. Find all ideals N of $\mathbb{Z}_{12}$ and in each case compute $\mathbb{Z}_{12}/N$.

17. If F is a field, show that every ideal in $F[x]$ is principal.

(6 x 2 = 12 weightage)

Part C: Answer *any two* questions. Each carries *5* weightage.

18. (a). Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

    (b). Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2, 3)\rangle$.

19. (a). If G is a finite group and p divides $|G|$, prove that the number of Sylow p-subgroups is congruent to 1 modulo p and divides $|G|$.

    (b). Show that $(a, b : a^5 = 1, b^2 = 1, ba = a^2b)$ represents a group of order 2.

20. (a). Let p be a prime. Let G be a finite group and let p divide $|G|$. Prove that G has an element of order p.

    (b). If H and K are finite subgroups of a group G, prove that $|HK| = \frac{(|H|)(|K|)}{|H \cap K|}$.

21. (a). Show that the polynomial $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$ for any prime p.

    (b). Prove that an ideal $\langle p(x)\rangle \neq \{0\}$ of $F[x]$ is maximal if and only if $p(x)$ is irreducible over F.

(2 x 5 = 10 weightage)