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D1AMT2205 Reg.No:.....

Name:.....

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2022

MATHEMATICS

FMTH1C05 - NUMBER THEORY

Time: 3 Hours Maximum Weightage: 30

PART A: Answer *all questions. Each carries* 1 *weightage*

1. Find all integers n such that $\phi(n) = \frac{n}{2}$.
2. Explain with an example that there exist multiplicative functions which are not completely multiplicative.
3. State and prove Selberg identity.
4. Prove that $[x] + \left[x + \frac{1}{2}\right] = [2x]$.
5. For $x > 0$, show that $\lim_{x\to\infty} \left(\frac{\psi(x)}{x} - \frac{\vartheta(x)}{x}\right) = 0$.
6. For all $x \geq 0$, prove that $\sum_{n\leq x} \frac{\Lambda(n)}{n} = xlogx + O(x)$.
7. Find the quadratic residues and non-residues modulo 11.
8. Define an affine crypto-system. Illustrate with an example.

(8 x 1 = 8 weightage)

Part B: Answer any *two questions from each unit. Each carries* 2 *weightage*

Unit 1

9. If f is multiplicative, then prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n)f(n)$ for all $n \geq 1$. Hence derive an expression for the inverse of Euler's ϕ function.

10. State and prove Euler's summation formula.

11. For $x \geq 2$, show that $\sum_{p \leq x} \left[\frac{x}{p}\right] logp = xlogx + O(x)$.

Unit 2

12. Prove that $\lim_{x\to\infty} \frac{\pi(x)logx}{x} = 1$ is logicaly equivalent to $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$.

13. If $\{a_n\}$ is a nonnegative sequence such that $\sum_{n \leq x} a(n) \left[\frac{x}{n}\right] = xlogx + O(x)$ for all $x \geq 1$, then prove that there is a constant B such that $\sum_{n \leq x} a(n) \leq Bx$ for all $x \geq 1$.

14. If $A(x) = \sum_{n \leq x} \frac{\mu(n)}{n}$, prove that the relation $A(x) = o(x)$ as $x \to \infty$ implies the prime number theorem.

Unit 3

15. State and prove the quadratic reciprocity law.

16. Define Jacobi symbol and prove that

$$(-1|P) = (-1)^{(P-1)/2}$$

and

$$(2|P) = (-1)^{(P^2-1)/8}$$

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17. In the 27-letter alphabet with 'blank = 26', use $A = \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \in M_2(\mathrm{Z}/26\mathrm{Z})$, to encipher the message unit "NO" assuming each plaintext message unit $P = \begin{bmatrix} x \\ y \end{bmatrix}$ is transformed into $C = \begin{bmatrix} x' \\ y' \end{bmatrix}$ by the rule C = AP.

(6 x 2 = 12 weightage)

Part C: Answer any *two questions. Each carries* 5 *weightage*

18. Show that the set of all arithmetical functions f with $f(1) \neq 0$ forms an a Abelian group with respect to the Dirichlet product.
19. Show that $\frac{1}{6}\frac{n}{logn} < \pi(n) < 6\frac{n}{logn}$ for every integer $n \geq 2$.
20. Prove that the prime number theorem implies

$$\lim_{x\to\infty} \frac{M(x)}{x} = 0.$$

21. State and prove Gauss' lemma.

(2 x 5 = 10 weightage)

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