D1AMT2202 **(2 Pages)** **Name....................**

Reg.No..................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2022

(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH1C02 - LINEAR ALGEBRA

Time: 3 Hours **Maximum weightage: 30**

Part A

Answer *all* questions. Each carries 1 weightage.

1. Let V be a vector space over $\mathbb{C}$. Suppose α, β and γ are linearly independent vectors in V. Prove that $(\alpha + \beta)$, $(\beta + \gamma)$ and $(\gamma + \alpha)$ are linearly independent.
2. Show that the inverse of a linear transformation, if it exists, is linear.
3. Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$. What is the matrix of T relative to the standard ordered basis of $\mathbb{R}^2$?
4. Find the dual basis of standard ordered basis of $\mathbb{R}^3$.
5. Show that similar matrices have the same characteristic polynomial.
6. Prove that if E is the projection on R along N, then $(I - E)$ is the projection on N along R.
7. Define inner product. Give an example.
8. Prove that an orthogonal set of non-zero vectors is linearly independent.

(8 x 1= 8 weightage)

Part B

Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Show that the subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.
10. Find the coordinate matrix of the vector $(1, 0, 1)$ relative to the ordered basis $\mathscr{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$ and $\alpha_3 = (0, -3, 2)$.
11. Let $T : V \to W$ be a linear transformation. Show that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

(P.T.O.)

Unit 2

12. Let V be a finite-dimensional vector space over the field F, and let W be a subspace of V. Show that $\dim\ W + \dim\ W^0 = \dim\ V$.

13. Let $g, f_1, \cdots, f_r$ be linear functionals on a vector space V with respective null spaces $N, N_1, \cdots, N_r$. Show that g is a linear combination of $f_1, \cdots, f_r$ if and only if N contains the intersection $N_1 \cap N_2 \cap \cdots \cap N_r$.

14. Let A be any $m \times n$ matrix over the field F. Show that the row rank of A is equal to the column rank of A.

Unit 3

15. Let $V = W_1 \oplus \cdots \oplus W_k$. Show that there exist k linear operators $E_1, \ldots, E_k$ on V such that
    (a) each E_i is a projection;
    (b) $E_i E_j = 0$,if $i \neq j$;
    (c) $I = E_1 + \cdots + E_k$;
    (d) the range of E_i is W_i.

16. Let V be an inner product space. Show that $|(\alpha \mid \beta)| \leq \|\alpha\|\|\beta\|$ and $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ for all α, β in V.

17. State and prove Bessel's inequality.

(6 x 2= 12 weightage)

Part C

Answer any *two* questions. Each carries 5 weightage.

18. (a) Suppose W_1 and W_2 are finite dimensional subspaces of a vector space V. Show that $dim\ W_1 + dim\ W_2 = dim\ (W_1 + W_2) + dim\ (W_1 \cap W_2)$.
    (b) Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each α in V there are unique vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

19. (a) Let V be a finite-dimensional vector space over F and let $\{\alpha_1, \cdots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \cdots, \beta_n$ be any vectors in W. Show that there is only one linear transformation T from V into W such that $T\alpha_j = \beta_j$ for $j = 1, \cdots, n$.
    (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $T(1, 0) = (1, 0, 0)$ and $T(0, 1) = (0, 0, 1)$. Find $T(2, 3)$.

20. Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Show that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \cdots (x - c_k)$ where $c_1, \cdots, c_k$ are distinct elements of F.

21. Apply the Gram-Schmidt process to the vectors $\beta_1 = (3, 0, 4), \beta_2 = (-1, 0, 7)$, $\beta_3 = (2, 9, 11)$, to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

(2 x 5= 10 weightage)