D1AMT2201 **(2 Pages)** **Name............................**
Reg.No..........................

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2022
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH1C01-ABSTRACT ALGEBRA

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Find all proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.
2. Find the order of the factor group $\mathbb{Z}_4 \times \mathbb{Z}_{12}/\langle(2,2)\rangle$.
3. Find the number of orbits in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ under the cyclic subgroup $\langle(1, 3, 5, 6)\rangle$ of S_8.
4. Let $G = \mathbb{Z}_{24}, H = \langle 4\rangle$and $N = \langle 6\rangle$. Compute HN.
5. Show that there are no simple groups of order 20.
6. Give isomorphic refinements of the two series $\{0\} < 10\mathbb{Z} < Z$and $\{0\} < 25\mathbb{Z} < Z$.
7. Find the sum and the product of the polynomials $f(x) = 4x - 5$ and $g(x) = 2x^2 - 4x + 2$ in the polynomial ring $\mathbb{Z}_8[x]$.
8. The polynomial $x^4 + 4$ can be factored into linear factors in $\mathbb{Z}_5[x]$. Find this factorization.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Find all abelian groups, up to isomorphism, of order 720.
10. Let H be a subgroup of a group G. Then prove that left coset multiplication is well defined by the equation $(aH)(bH) = (ab)H$ if and only if H is a normal subgroup of G.
11. Let X be a G- set, where G is a finite group and let $x \in X$. Prove that $|Gx| = (G: G_x)$ and hence $|Gx|$ is a divisor of $|G|$.

Unit 2

12. If N is a normal subgroup of G, and if H is any subgroup of G, then prove the following
    a) $H \vee N = H N = NH$.
    b) If H is also normal in G, then HN is normal in G.
13. Let P_1 and P_2 be Sylow$p$-subgroups of a finite group G. Prove that P_1 and P_2 are conjugate subgroups of G.

(P.T.O.)

14. Let G be a group of order 36 and let H be a nontrivial normal subgroup of G. Prove that $H = G$.

Unit 3

15. State and prove Eisenstein Criterion.

16. Let R be a ring with unity. If R has characteristic $n > 1$, then prove that R contains a subring isomorphic to $\mathbb{Z}_n$. Is it possible that a ring with unity may simultaneously contain two subrings isomorphic to $\mathbb{Z}_n$ and $\mathbb{Z}_m$ for $n \neq m$? If it is possible, give an example. If it is impossible, prove it.

17. Prove that the quaternions $\mathbb{H}$ form a strictly skew field under addition and multiplication.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. a) Prove that M is maximal normal subgroup of G if and only if G/M is simple.

    b) Find the number of distinguishable ways the edges of a square of cardboard can be painted if six colours of paint are available and

    i) no colour is used more than once.
    ii) the same colour can be used on any number of edges.

19. Let G be a group. The set of all commutators $aba^{-1}b^{-1}$ for $a, b \in G$ generates a subgroup C (the commutator subgroup) of G.
    a) Prove the following.
    i) This subgroup C is a normal subgroup of G.
    ii) If N is a normal subgroup of G, then G/N is abelian if and only if $C \leq N$.
    b) Determine all groups of order 10 up to isomorphism.

20. a) Let $\{H_i\}$ and $\{K_i\}$ be two composition series of a group G. Prove that $\{H_i\}$ and $\{K_i\}$ are isomorphic.

    b) Let $\{H_i\}$ be a composition series of a group G and let N be a proper normal subgroup of G. Prove that there exists a composition series containing N.

    c) Is the group S_3 solvable?

21. a) Prove that every ideal in $F[x]$ is principal, where F is a field.

    b) Prove that if the ideal $\langle p(x) \rangle \neq \{0\}$ of $F[x]$ is maximal then $p(x)$ is irreducible over F, where F is a field.

    c) Find all $c \in \mathbb{Z}_5$ such that $\mathbb{Z}_5[x]/x^2 + x + c$ is a field.

(2 × 5= 10 weightage)