FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2021
(Regular/Improvement/Supplementary)

STATISTICS
FMST1C02-ANALYTICAL TOOL FOR STATISTICS I

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer any *four* questions. Each carries *two* weightage.

1. Define countable and uncountable sets. Give one example for each.
2. Show that the intersection of an arbitrary family of closed sets is closed.
3. (i) Find $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$

   (ii) Find the derivative of the function $f(x) = \begin{cases} x & if \quad 0 \le x < 1 \\ 1 & if \quad x \ge 1 \end{cases}$ at x= 1.
4. Explain the following: (i) Connectedness and (ii) Compactness.
5. Define Riemann Stieltjes integral of a function f(x) with respect to α. Using this method, find the value of $\int_0^3 [x]\, d(x^2)$, $(where\ [x]\ is\ the\ integer\ part\ of\ \ x)$.
6. Define pointwise and uniform convergence of a sequence of functions. State the mutual relationships between these two.
7. Discuss the uniform convergence of the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}$ on a closed interval [a,b].

(4 × 2 = 8 weightage)

Part B: Answer any *four* questions. Each carries *three* weightage

8. Define a metric space. Verify whether the function $d: R \times R \to R$, is a metric on R, where d$(x, y) = \ |x - y|, for\ all\ x, y\ \in R$.
9. Prove that continuous image of a connected set is connected.
10. (i)Prove or disprove: If the derivative f'exists and monotonic on (a,b), then f' is continuous on (a,b).

    (ii) Distinguish between continuity and uniform continuity of real functions.
11. State Lagrange's mean value theorem. Examine the theorem for the function $f(x) = x(x-1)(x-2)$ for $0 \le x \le 1/2$.

12. Prove the following:

If $f \in R(\alpha)$ on [a,b], then $|f| \in R(\alpha)$ on [a,b] and $\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\ d\alpha(x)$.

13. If $f \in R(\alpha)$ on [a,b], prove that $cf \in R(\alpha)$ on [a,b] where c is a constant.

14. Show that $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ converges uniformly on [0,a] for any a>0.

(4 × 3 = 12 weightage)

Part C: Answer any *two* questions. Each carries *five* weightage.

15. (i) Show that every closed subset of a compact metric space is compact.

(ii) Prove that every infinite bounded set has a limit point.

16. (i) Check the continuity of the function $f(x) = \begin{cases} 2(x-1) & if \quad 0 < x < 1 \\ 2x & if \quad x \geq 1 \end{cases}$ at x=1

(ii) Show that if f is derivable at c and $f(c) \neq 0$, then $\frac{1}{f}$ is also derivable and obtain the derivative.

(iii) Obtain the Maclaurin series expansion of cos x, using higher order derivatives.

17. (i) Prove that, a function f is integrable with respect to α if and only if for every $\epsilon > 0$, there exists a partition P of [a,b], such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

(ii) Discuss the effect of refinement of a partition on the upper and lower Riemann Stieltjes sums with proper reasons.

18. (i) Prove: If a series $\sum f_n$ converges uniformly to f on [a,b] and the terms f_n are continuous at a point in the interval, then f is also continuous at that point.

(ii) Show that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots$ is not uniformly convergent on [0,1].

(2 × 5 = 10 weightage)