D1AMT2105

Reg. No..............................
Name: ....

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2021
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH1C05 - NUMBER THEORY

Time: Three Hours Max. Weightage : 30

Part A

Answer **all** questions.
Each question carries 1 weightage.

1. Define multiplicative function. Give example of a function which is multiplicative but not completely multiplicative.
2. Define derivative of an arithmetical function and show that $(f * g)' = f' * g + f * g'$
3. State Legendre's identity and find the number of zeros at the end of 1000!.
4. Define Chebyshev's ψ function and express in terms of logarithm.
5. Show that congruence is an equivalence relation.
6. Solve the congruence $25x \equiv 15(\mod 120)$
7. Determine whether 888 is a quadratic residue or non residue of the prime 1999.
8. Define cryptosystem. Give example.

(8 x 1 = 8 Weightage)

Part B

Answer any **two** questions from each unit.
Each question carries 2 weightage.

Unit I

9. If f is multiplicative, State and prove the necessary and sufficient condition for f to be completely multiplicative.
10. State and prove Euler's summation formula.
11. State and prove Selberg identity.

Unit II

12. State and prove Abel's identity.

13. Show that for $n \geq 1$ the nth prime p_n satisfies the inequalities

$$\frac{1}{6} n \log n < p_n < 12(n \log n + n \log \frac{12}{e})$$

14. Show that for any prime p the coefficients of the polynomial

$$f(x) = (x-1)(x-2)\cdots(x-p+1) - x^{(p-1)} + 1$$

are divisible by p and hence deduct Wilson's theorem.

Unit III

15. Show that for every odd prime p we have

$$(2|p) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv \pm 1 (mod\ 8), \\ -1 & \text{if } p \equiv \pm 3 (mod\ 8) \end{cases}$$

16. Determine odd primes for which 3 is a quadratic residue and those for which it is a non residue.

17. Define Affine cipher. In the 26 letter alphabet, use the affine enciphering transformation with key a = 13, b = 9 to encipher the message "MATH".

(6 x 2 = 12 Weightage)

Part C

Answer any **two** questions. Each carries 5 weightage

18. Show that for $x \geq 2$, $\sum_{p \leq x} [\frac{x}{p}] \log p = x \log x + O(x)$.

19. State Chinese remainder theorem. And find all x which simultaneously satisfy the system of congruences, $x \equiv 1 (mod\ 3)$, $x \equiv 2 (mod\ 4)$, $x \equiv 3 (mod\ 5)$

20. State and prove Quadratic reciprocity law for Legendre symbol.

21. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/\mathbb{N}\mathbb{Z})$ and $D = ad - bc$. Show that the following are equivalent.

    a. g.c.d$(D, \mathbb{N}) = 1$
    b. A has an inverse matrix
    c. if x and y are not both 0 in $\mathbb{Z}/\mathbb{N}\mathbb{Z}$, then $A \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
    d. A gives a 1-to-1 correspondence of $(\mathbb{Z}/\mathbb{N}\mathbb{Z})^2$ with itself

(2 x 5 = 10 Weightage)