D1AMT2102 **(Pages:2)** **Name: ....**

Reg. No.: ....

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2021
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH1C02 - LINEAR ALGEBRA

Time: 3 **hours** **Maximum weightage:** 30

PART A

Answer **all** questions. Each question carries 1 weightage.

1. If F is a field, then verify that F^n is a vector space over the field F.
2. Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.
3. Let V be the real vector space of all polynomial functions from R into R of degree 2 or less, i.e., the space of all functions f of the form
$$f(x) = c_0 + c_1x + c_2x^2.$$
Let t be a fixed real number and define
$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2.$$
Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If $f(x) = c_0 + c_1x + c_2x^2$, what are the coordinates of f in this ordered basis $\mathcal{B}$?
4. Define a dual space of a vector space. Prove that the dimensions of finite dimensional vector space and its dual space are equal.
5. Justify that similar matrices have the same characteristic polynomial.
6. State Cayley-Hamilton theorem.
7. Let $W_1, \ldots, W_k$ be subspaces of the vector space V. When do we say that $W_1, \ldots, W_k$ are independent?
8. Define inner product on a vector space V.

(8 x 1= 8 weightage)

PART B

Answer any **two** questions from each unit. Each question carries 2 weightage.

UNIT I

9. Show that the vectors $\alpha_l = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of α_l, α_2 and α_3.
10. Let T be a linear transformation from V into W. Prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.
11. Define isomorphism of vector spaces V onto W. Prove that every $n-$dimensional vector space over the field F is isomorphic to the space F^n.

UNIT II

12. Let T be the linear operator on $\mathbb{R}^3$ defined by
$T(x_1x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$.
Find the matrix of T in the standard ordered basis for $\mathbb{R}^3$?

13. Prove that the trace function is a linear functional on the matrix space F^{nXn}.

14. Let V be a finite-dimensional vector space over the field F, and let W be a subspace of V and W^0 be the annihilator of W. Then show that $\dim W + \dim W^0 = \dim V$.

UNIT III

15. Let V be a finite-dimensional vector space and let $W_1, \ldots, W_k$ be subspaces of V such that $V = W_1 + \ldots + W_k$ and $\dim V = \dim W_1 + \ldots + \dim W_k$.
Prove that $V = W_1 \oplus \ldots \oplus W_k$.

16. Show that an orthogonal set of non-zero vectors is linearly independent.

17. Prove that

    (a) $|(\alpha|\beta)| \leq ||\alpha|| \, ||\beta||$ and

    (b) $||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$.

(6 x 2= 12 weightage)

PART C

Answer any **two** questions. Each question carries 5 weightage.

18. Let V be an $n-$dimensional vector space over the field F, and let W be an $m-$ dimensional vector space over F. Then, show that the space $L(V, W)$ is finite-dimensional and has dimension mn.

19. If f is a non-zero linear functional on the vector space V, then prove that the null space of f is a hyperspace in V. Also show that every hyperspace in V is the null space of a non-zero linear functional on V.

20. Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \ldots (x - c_k)$ where $c_1, \ldots, c_k$ are distinct elements of F.

21. Let V be an inner product space and let $\{\beta_1, \ldots, \beta_n\}$ be any independent vectors in V. Then construct orthogonal vectors $\alpha_1, \ldots \alpha_n$ in V using Gram-Schmidt orthogonalization process so that $\{\alpha_1, \ldots \alpha_n\}$ is a basis for the subspace spanned by $\beta_1, \ldots, \beta_n$.

(2 x 5= 10 weightage)