D1AMT2101 **(2 Pages)** **Name............................**
Reg.No..........................

FIRST SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2021
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH1C01-ABSTRACT ALGEBRA

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each of the elements. Is this group cyclic?
2. Find the order of the factor group $\frac{(\mathbb{Z}_{11}\times\mathbb{Z}_{15})}{\langle(1,1)\rangle}$.
3. Find the number of orbits in $\{1,2,3,4,5,6,7,8\}$ under the cyclic subgroup $(1,3,5,6)$ of S_8.
4. Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ be the homomorphism such that $\phi(1) = 2$. Find the kernelof ϕ.
5. Give isomorphic refinements of the two series,

   $\{0\} < 10\,\mathbb{Z} < \mathbb{Z}$and $\{0\} < 25\,\mathbb{Z} < \mathbb{Z}$.

6. Prove that every group of order $(5)(7)(47)$ is abelian and cyclic.
7. Find the sum and the product of the polynomials

   $f\,(x) = 4x - 5\ and\ \ g(x) = 2x^2 - 4x + 2$ in the polynomial ring $\mathbb{Z}_8[x]$.

8. The polynomial $x^4 + 4$ can be factored into linear factors in $\mathbb{Z}_5[x]$. Find this factorization.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Find

   a) the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$.

   b) all abelian groups of order 1089, up to isomorphism.

10. Let X be a G-set. For each $g \in G$, the function $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$ for $x \in X$. Show that σ_g is a permutation of X. Also, show that the map $\phi : G \to S_X$ defined by $\phi(g) = \sigma_g$ is a homomorphism with the property that $\phi(g)(x) = gx$.
11. Prove that M is a maximal normal subgroup of G if and only if $\frac{G}{M}$ is simple.

(P.T.O.)

Unit 2

12. State and prove first Sylow Theorem.

13. If p and q are distinct primes with $p < q$, then every group G of order pq has a single subgroup of order q and this subgroup is normal in G. Hence G is not simple. If q is not congruent to 1 modulo p, then prove that G is abelian and cyclic.

14. Determine all groups of order 10 up to isomorphism.

Unit 3

15. Prove that a nonzero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in the field F.

16. Prove that the quaternions H form a strictly skew field under addition and multiplication.

17. Let H be a subring of the ring R. Prove that multiplication of additive cosets of H is well defined by the equation $(a + H)(b + H) = ab + H$ if and only if $ah \in H$ and $hb \in H$ for all $a, b \in R$ and $h \in H$.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. a) The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

    b) Let G be a finite group and X a finite G −set. If r is the number of orbits in X under G, then prove that

$$r \cdot |G| = \sum_{g \in G} |X_g|.$$

19. a) If H and K are finite subgroups of a group G, then prove that $|HK| = \frac{(|H|)(|K|)}{|H \cap K|}$.

    b) Show that every group of order 255 = (3)(5)(17) is abelian.

20. a) Let G be generated by $A = \{a_i \mid i \in I\}$ and let G' be any group. If $a_i{}'$ for $i \in I$ are any elements in G', not necessarily distinct, then prove that there is at most one homomorphism $\phi : G \to G'$ such that $\phi(a_i) = a_i'$.

    b) Show that $(x, y : y^2x = y, yx^2y = x)$ is a presentation of the trivial group of one element.

21. Let R be a commutative ring with unity. Then prove that M is a maximal ideal of R if and only if $\frac{R}{M}$ is a field.

(2 × 5 = 10 weightage)